Guessing clubs for aD, non D-spaces

Dániel Soukup

Eötvös Loránd University, Institute of Mathematics

Guessing sequences

Let $\lambda > \mu = cf(\mu)$ and $S^\lambda_\mu = \{\alpha \in \lambda : cf(\alpha) = \mu\}$.

Definition (Ostaszewski)

A sequence $\{A_\alpha : \alpha \in S^{\omega_1}_\omega\}$ of subsets of ω_1 is a ♣-sequence iff

- *A_α is a cofinal ω-type sequence in α, and*
- *for all $A \in [\omega_1]^{\omega_1}$ there is some $\alpha \in S^{\omega_1}_\omega$ such that $A_\alpha \subseteq A$.*

We say that ♣ **holds** iff there is a ♣-sequence.

- very useful in **constructive** proofs (set-theory, topology)
- ♣ is **independent from ZFC**
- **many variations** (Jensen's $\diamondsuit$, Juhasz's axiom (t), ...)

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An S^λ_μ***-club sequence*** *is a sequence* $\underline{C} = \langle C_\alpha : \alpha \in S^\lambda_\mu \rangle$ *such that* $C_\alpha \subseteq \alpha$ ***is a club in*** α *of order type* μ*.*

Theorem (Shelah)

Let λ *be a cardinal such that* $cf(\lambda) \geq \mu^{++}$ *for some regular* μ*. Then there is an* S^λ_μ***-club sequence*** $\underline{C} = \langle C_\alpha : \alpha \in S^\lambda_\mu \rangle$ *such that* ***for every club*** $E \subseteq \lambda$ ***there is*** $\alpha \in S^\lambda_\mu$ *(equivalently, stationary many) such that* $C_\alpha \subseteq E$*.*

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Let $\mu = cf(\mu) > \omega$ and take any $S^{\mu^+}_\mu$-club sequence $\underline{C} = \langle C_\alpha : \alpha \in S^{\mu^+}_\mu \rangle$ such that $C_\alpha = \{a^\xi_\alpha : \xi < \mu\} \subseteq \alpha$.

For every club $E \subseteq \lambda$, there is $\alpha \in S^{\mu^+}_\mu$ such that

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- compact spaces

Definition

*An **open neighborhood assignment** (ONA, in short) on a space (X,τ) is a map $U : X \to \tau$ such that $x \in U(x)$ for every $x \in X$.*

X is **compact** $\Leftrightarrow$ for every ONA U on X there is a **finite** $D \subseteq X$ such that $X = \bigcup U[D]$

- generalization: **finite** $D \longrightarrow$ **locally finite** D.

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D-spaces

Definition (E. van Douwen)

*X is a **D-space** iff for every neighborhood assignment U, there is a **closed and discrete** $D \subseteq X$ (i.e. locally finite) such that $X = \bigcup U[D]$.*

- every σ-compact or metric space is a D-space
- ω_1 **is not a D-space** (every closed discrete set is finite, however non compact)

Problem (E. van Douwen)

*Is it true, that **every Lindelöf space is a D-space**?*

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D-spaces and aD-spaces

(X, τ) D-space $\Leftrightarrow$ iff for every ONA U, there is a closed and discrete $D \subseteq X$ such that $X = \bigcup U[D]$.

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*A **cover** $\mathcal{U}$ of a space X is **irreducible** iff there is no proper subcover of $\mathcal{U}$.*

Definition (Arhangel'skii, 2002)

*A space X is an **aD-space** iff **for every closed $F \subseteq X$ and open cover $\mathcal{U}$ of F** there is an **irreducible open refinement of $\mathcal{U}$**.*

- irreducible open cover $\leftrightarrow$ ONAs on closed discrete sets

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(X,τ) is a D-space $\Rightarrow$ (X,τ) is an aD-space.

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There exists *an aD, non D-space.*

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The general construction

We need the following parameters:

- infinite cardinals $\lambda > \mu = cf(\mu)$,
- a MAD family $\mathcal{M} \subseteq [\mu]^\mu$, enumerated as $\mathcal{M} = \{M^\varphi : \varphi < \kappa\}$,
- an S^λ_μ-club sequence $\underline{C} = \{C_\alpha : \alpha \in S^\lambda_\mu\}$.

We define a space $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$ on a subset of $\lambda \times \kappa$.

- $X_\alpha = \{(\alpha, 0)\}$ for $\alpha \in \lambda \setminus S^\lambda_\mu$,
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Fix $\lambda > \mu = cf(\mu)$, a MAD family $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$ and S^λ_μ-club sequence $\underline{C}$.

Claim

The space $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is 0-dimensional, T_2 and scattered. For any $A \in [\lambda]^{<\mu}$ the set $\bigcup\{X_\alpha : \alpha \in A\}$ is closed discrete.

Let $\pi(F) = \{\alpha < \lambda : X_\alpha \cap F \neq \emptyset\}$ for any $F \subseteq X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$.

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X is an aD-space $\Leftrightarrow$ for every closed $F \subseteq X$ and open cover $\mathcal{U}$ of F there is an irreducible open refinement of $\mathcal{U}$.

Definition

*Let $F_\alpha = F \cap X_\alpha$ for $F \subseteq X$ and $\alpha < \lambda$. A subset $F \subseteq X$ **is high enough** if*

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The space X is high iff every closed, unbounded $F \subseteq X$ is high enough.

Main Theorem

*If $cf(\lambda) \geq \mu$ and $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$ **is high**, then X **is an** aD**, non** D**-space**.*

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Guessing clubs for high $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ spaces

Shelah: if $cf(\lambda) \geq \mu^{++}$ for some regular μ then there is an S^λ_μ-club sequence such that for every club E there is stationary many $\alpha \in S^\lambda_\mu$ such that $C_\alpha \subseteq E$.

Claim

*If $C_\alpha \subseteq \pi(F)'$ **for a closed** $F \subseteq X$ and $\alpha \in S^\lambda_\mu$, then $F_\alpha = X_\alpha$.*

Corollary

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Corollary

$2^\omega \geq \omega_2$ *Let $\mathcal{M}$ be a MAD family on ω of size 2^ω and let $\underline{C}$ be an $S^{\omega_2}_\omega$-club guessing sequence from Shelah. Then $X[\omega_2, \omega, \mathcal{M}, \underline{C}]$ is high.*

$2^\omega = \omega_1$ and $2^{\omega_1} \geq \omega_3$ *Let $\mathcal{M}$ be a MAD family on ω_1 of size 2^{ω_1} and let $\underline{C}$ be an $S^{\omega_3}_{\omega_1}$-club guessing sequence from Shelah. Then $X[\omega_3, \omega_1, \mathcal{M}, \underline{C}]$ is high.*

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If $cf(\lambda) \geq \mu^{++}$ for some regular μ, $\underline{C}$ is an S^{λ}_{μ}-club guessing sequence from Shelah, $\mathcal{M}$ is a MAD on μ of size at least $\lambda \Rightarrow X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is high.

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$2^{\omega} \geq \omega_2$ *Let $\mathcal{M}$ be a MAD family on ω of size 2^{ω} and let $\underline{C}$ be an $S^{\omega_2}_{\omega}$-club guessing sequence from Shelah. Then* $X[\omega_2, \omega, \mathcal{M}, \underline{C}]$ ***is high.***

$2^{\omega} = \omega_1$ **and** $2^{\omega_1} \geq \omega_3$ *Let $\mathcal{M}$ be a MAD family on ω_1 of size 2^{ω_1} and let $\underline{C}$ be an $S^{\omega_3}_{\omega_1}$-club guessing sequence from Shelah. Then* $X[\omega_3, \omega_1, \mathcal{M}, \underline{C}]$ ***is high.***

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$2^{\omega_1} = \omega_2$ *Let $\underline{C}$ be an $S^{\omega_2}_{\omega_1}$-club guessing sequence from Shelah and let $\mathcal{M}_{NS}$ be a nonstationary MAD family on ω_1. Then $X[\omega_2, \omega_1, \mathcal{M}_{NS}, \underline{C}]$* ***is high****.*

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Thank you for your attention...

... and I would be happy to answer any questions!